

Dark Energy and the Schwarzian Derivative

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Abstract

Theories with a time dependent Newton's constant admit two natural measures of time : atomic and astronomical. Temporal parametrisation by $SL(2, \mathbb{R})$ transformations gives rise to an equivalence between theories with different time dependence's, including the special Case of no time dependence, a fact noticed by Mestschersky, Vinti and by Lynden-Bell. I point out that theories with time dependent dark energy densities admit three natural measures of time : atomic and astronomical and de Sitter related by temporal re-parametrizations and I extend Mestschersky-Vinti-Lynden-Bell's result to cover this more general situation. I find a consequent equivalence between theories in which the density of dark energy is constant in time and in which it varies with time. Strikingly a time dependent cosmological constant changes by the addition of a Schwarzian derivative term unless the temporal reparameterization belongs to $SL(2, \mathbb{R})$. In General Relativity one may introduce a Schwarzian tensor to investigate how the notion of dark energy changes under changes of conformal frame. The general theory is illustrated in the case of Friedmann-Lemaitre metrics.

1 Introduction

Until the development of Quartz, Ammonia and Caesium clocks, time measurements were astronomical, and the default assumption was that with respect to those units, Newton's law of gravity was independent of time. The most economical assumption was then that the rate of atomic processes are governed by the same units [1]. Thus the times which enter Kepler's law and Schrödinger's equation are the same and coincide with those that enter Maxwell's equations. In which case the three "fundamental constants of physics" G , \hbar and c would indeed be constants. and (Planck) units could be adopted in which they be taken without loss of generality to equal unity [2].

However the constancy of all three constants has been questioned ¹, most notably G by, among others, Dirac [3], Jordan [4], Brans and Dicke [5]. One may also question the constancy of \hbar and c but the evidence against any time variability appears to be so strong that in this paper I shall assume that they are indeed constant. This and the evidence in favour of the Weak Equivalence Principle is usually taken to justify ones belief in a spacetime metric $g_{\mu\nu}$ such that proper time along the world line of an idealised observer or experimenter coincides with atomic time and more generally the time of the standard model. In principle, the other various constants of the standard model, could vary with time, but current limits appear to be extremely stringent and so in this paper I shall assume that such things as the “fine structure constant”² are indeed constant. If this is not true, we would for example, have to introduce Stoney time [6, 7].

While the precision of Einstein’s equations have now been tested to an impressive level of accuracy, limits on the time dependence of Newton’s constant G over cosmological times are not as stringent and moreover the discovery of cosmic acceleration [8, 9] quantified by an effective cosmological ³ constant Λ has suggested that one should question the constancy in time of both. This could arise, for example if both G and Λ depend locally on one or more scalar fields ϕ and contribute (in which units chosen so that $c = 1$) to the action a term

$$\frac{1}{16\pi} \int \left(\frac{R}{G(\phi)} - 2\Lambda(\phi) \right) \sqrt{-g} + \dots \quad (1)$$

where the ellipsis denotes the contribution from the standard model and the scalar ϕ . By means of a suitable Weyl conformal rescaling

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu} \quad (2)$$

the action becomes

$$\frac{1}{16\pi} \int \left(\frac{\tilde{R}}{\tilde{G}(\phi)} - 2\tilde{\Lambda}(\phi) \right) \sqrt{-\tilde{g}} + \dots \quad (3)$$

By choosing $\Omega(\phi)$ appropriately one may arrange that $\tilde{G} = \text{constant}$, or $\tilde{\Lambda} = \text{constant}$ but in general not both. It is customary to refer to the metric $g_{\mu\nu}$ as the Jordan conformal frame and that metric $\tilde{g}_{\mu\nu}$ for which $\tilde{G} = \text{constant}$ as the Einstein conformal frame. Therefore it seems not unreasonable to refer to that

¹A linguistic purist might justifiably object to this oxymoron and even point out that it only makes meaningful physical sense to say that dimensionless ratios of physical quantities may vary with time. However since the construction of the requisite dimensionless quantities is little more than an elementary undergraduate exercise I shall not trouble the reader by spelling it out in detail. Thus for example in what follows “time dependence of Newton’s constant” should be understood in the same sense that one understands the generally accepted usage of the phrase “time dependence of Hubble’s constant”.

²i.e. Sommerfelds’s constant

³Perhaps with more justice nowadays this should be called the de Sitter’s constant

metric $\tilde{g}_{\mu\nu}$ for which $\tilde{\Lambda} = \text{constant}$ as the De-Sitter conformal frame. Since the Einstein conformal frame metric may be said to measure Planck units, one may also say that De-Sitter conformal frame measures De-Sitter units.

In this paper I shall show that under temporal re-parameterisations, or more generally Weyl conformal rescaling, the cosmological constant changes by the addition of a term involving the Schwarzian derivative, or more generally the Schwarzian tensor of the transformation. In more detail, the plan of the paper is as follows. In section 2 I recall some elementary, but little noticed, facts about temporal reparametrisations in classical mechanics. showing how potentials change by the addition of a Schwarzian derivative term and I relate this to a conformal rescaling of the associated higher dimensional Bargmann metric from which the equations of motion may be obtained by means of a null reduction. In section 3 this general theory is applied to a Newtonian cosmological model. In particular I show that the cosmological constant or equivalently of dark energy changes by a Schwarzian derivative term under temporal reparametrisations. In section 4 I turn to the General Relativistic theory and the behaviour under conformal transformations showing that Schwarzian tensor enters. In section 5 I apply these results to the Friedmann-Lemaître cosmologies and in particular to the Λ CDM model both at the Newtonian and General Relativistic levels. I note a connection between the so-called cosmological scalars including acceleration and jerk and the Schwarzian derivative. Section 6 is a short conclusion in which I suggest it would be interesting to see whether or how the Schwarzian derivative enters the transformation formulae for the energy momentum tensor in 3+1 dimensional CFT's.

2 Temporal Re-parameterisations

We begin by placing the old results of Mestschersky [10] Vinti [11] and Lynden-Bell [12] in a more general context and extend them to the case of a cosmological constant. Our starting point will be at the Newtonian level. Later we will make contact with Einstein's covariant viewpoint.

Consider a system of N point particles of mass m_a and position vector \mathbf{x}_a moving in \mathbb{E}^3 and governed by the action

$$\int \left\{ \sum_{1 \leq a \leq N} \frac{1}{2} m_a \dot{\mathbf{x}}_a^2 - U(\mathbf{x}_a, t) \right\} dt, \quad (4)$$

where $\dot{}$ signifies differentiation with respect to t . If we make the replacements

$$t = f(\tilde{t}), \quad \mathbf{x}_a = \sqrt{f'} \tilde{\mathbf{x}}_a \quad (5)$$

where $\tilde{}$ denotes differentiation with respect to \tilde{t} we find, up to a boundary term, that the action becomes

$$\int \left\{ \sum_{1 \leq a \leq N} \frac{1}{2} m_a \tilde{\mathbf{x}}_a'^2 - \tilde{U}(\tilde{\mathbf{x}}_a, \tilde{t}) \right\} d\tilde{t} \quad (6)$$

where

$$\tilde{U}(\tilde{\mathbf{x}}_a, \tilde{t}) = f'U(\sqrt{f'}\tilde{x}_a, f(\tilde{t})) + \frac{1}{4}\{f, \tilde{t}\} \sum_{1 \leq a \leq N} m_a \tilde{x}_a^2 \quad (7)$$

and

$$\{f, \tilde{t}\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \quad (8)$$

is the Schwarzian derivative⁴ of f with respect to \tilde{t} .

It is illuminating to view the transformation (5) as a diffeomorphism acting on the Eisenhart lightlike lift [15, 16, 17, 18, 20, 21, 22, 23] of the Lagrangian system (4) to a system of null geodesics of the Bargmann metric on $\mathbb{E}^{3N} \times \mathbb{R}^2$ given by

$$ds_B^2 = \sum_{1 \leq a \leq N} m_a d\mathbf{x}_a^2 + 2dt ds + 2U(\mathbf{x}_a, t) dt^2 \quad (9)$$

with t an affine parameter along the null geodesics Null geodesics (but not their affine parametrisation) are well known to be independent of Weyl conformal rescaling of the metric and we note that substituting (5) in (9) gives

$$ds_B^2 = f' \left\{ \sum_{1 \leq a \leq N} m_a d\tilde{\mathbf{x}}_a^2 + 2d\tilde{t} d\tilde{s} + 2\tilde{U}(\tilde{x}_a, \tilde{t}) d\tilde{t}^2 \right\} \quad (10)$$

where

$$\tilde{s} = s - \frac{1}{2} \frac{f''}{f'} \sum_{1 \leq a \leq N} m_a \tilde{x}_a^2. \quad (11)$$

The metric (9) and the metric inside the brace in (10) are therefore conformal but are not in general isometric and so (5) is in general not a conformal isometry⁵ However, if $U = 0$ and

$$\{f, \tilde{t}\} = 0, \quad \Longleftrightarrow \quad f = \frac{A\tilde{t} + B}{C\tilde{t} + D}, \quad AD - BC \neq 0, \quad (12)$$

we obtain the Moebius or fractional linear subgroup $PSL(2, \mathbb{R})$ of proper conformal symmetries of the 13-dimensional non-relativistic conformal group of a system of free non-relativistic particles. In particular if $B = C = 0$, $f' = A^2$ and (5) becomes

$$t = A^2 \tilde{t}, \quad \mathbf{x}_a = A \tilde{\mathbf{x}}_a. \quad (13)$$

Note that because t and \mathbf{x}_a scale in different ways, non-relativistic conformal symmetries do not preserve the speed of light and are quite distinct from relativistic conformal symmetries as we shall see in detail in §5.

⁴The notation is due to Cayley [14] and should not be confused with that for Poisson brackets. Schwarz originally used $\psi((f, t)$, Klein $[f]_t$ and Koebe $D(f)_t$. More recently $S(f)$ has become common.

⁵An earlier related observation on a time dependent oscillator, but with no reference the Schwarzian derivative may be found in [24].

3 Newtonian Cosmology

At the Newtonian level we choose [17, 18, 19]

$$U(\mathbf{x}_a, t) = -G(t) \sum_{1 \leq a < b \leq N} \frac{m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|} - \frac{\Lambda(t)}{6} \sum_{1 \leq a \leq N} m_a \mathbf{x}_a^2. \quad (14)$$

Thus

$$\tilde{U}(\tilde{\mathbf{x}}_a, \tilde{t}) = -\tilde{G}(\tilde{t}) \sum_{1 \leq a < b \leq N} \frac{m_a m_b}{|\tilde{\mathbf{x}}_a - \tilde{\mathbf{x}}_b|} - \frac{\tilde{\Lambda}(\tilde{t})}{6} \sum_{1 \leq a \leq N} m_a \tilde{\mathbf{x}}_a^2, \quad (15)$$

where

$$\tilde{G}(\tilde{t}) = \sqrt{f'} G(f(\tilde{t})) \quad (16)$$

$$\tilde{\Lambda}(\tilde{t}) = (f')^2 \Lambda(f(\tilde{t})) - \frac{3}{2} \{f, \tilde{t}\}. \quad (17)$$

While temporal re-parameterisations keep us within the class of models with a time dependent Newtonian attraction and time dependent cosmic repulsion, if we start with $\Lambda(t) = 0$, we shall find that in general a time dependent cosmological term is induced. The exceptional case is, as explained in [17], when

$$\{f, \tilde{t}\} = 0, \quad \Longleftrightarrow \quad f = \frac{A\tilde{t} + B}{C + D\tilde{t}}, \quad AD - BC \neq 0, \quad (18)$$

and coincides with that originally considered by Mestschersky [10], Vinti [11] and Lynden-Bell [12] (see also Barrow [13]) who noted that if we assume that $G(t) = \text{constant}$, we may obtain an example of a time dependent Newton's constant all of whose solutions may be obtained from those with a time-independent Newton's constant.

We could ask whether starting from a case in which $\Lambda(t)$ and $G(t)$ are constant we could obtain a case with $\tilde{\Lambda} = 0$. This would entail solving the equation

$$\frac{2}{3} (f')^2 \Lambda = \{f, \tilde{t}\} = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2. \quad (19)$$

Setting

$$y = \frac{f''}{(f')^2} = -\frac{1}{2} \left(\frac{1}{f'} \right)', \quad (20)$$

we start with

$$\frac{2}{3} \Lambda = y' = \frac{3}{2} y^2, \quad (21)$$

whence

$$\frac{3}{2\Lambda} \frac{dy}{1 - (\frac{3y}{2\sqrt{\Lambda}})^2} = d\tilde{t}. \quad (22)$$

Thus

$$y = \frac{2}{3} \sqrt{\Lambda} \tanh(\sqrt{\Lambda}(\tilde{t} - \tilde{t}_0)) = \frac{2}{3} \left(\ln(\cosh(\sqrt{\Lambda}(\tilde{t} - \tilde{t}_0))) \right)', \quad (23)$$

and hence

$$\frac{2}{3} \ln \left(\cosh \sqrt{\Lambda} (\tilde{t} - \tilde{t}_0) \right) = A - \frac{1}{2(f')^2}, \quad (24)$$

where \tilde{t}_0 and A are constants of integration.

3.1 CFT

The formula for the change of the cosmological constant under a temporal reparametrisations may be written as

$$-\frac{2}{3} \tilde{\Lambda}(\tilde{t}) = -\frac{2}{3} \Lambda(t) \left(\frac{dt}{d\tilde{t}} \right)^2 + \{t, \tilde{t}\}. \quad (25)$$

or in terms of “quadratic differentials”.

$$-\frac{2}{3} \tilde{\Lambda}(\tilde{t}) (d\tilde{t})^2 = -\frac{2}{3} \Lambda(t) (dt)^2 + \{t, \tilde{t}\} (d\tilde{t})^2. \quad (26)$$

The asymmetry in (25) and (26) is only apparent since [14])

$$\{t, \tilde{t}\} = -\{\tilde{t}, t\} \left(\frac{dt}{d\tilde{t}} \right)^2, \quad \Longleftrightarrow \quad \{t, \tilde{t}\} (d\tilde{t})^2 = -\{\tilde{t}, t\} (dt)^2. \quad (27)$$

and so we have

$$\left(\frac{2}{3} \tilde{\Lambda}(\tilde{t}) (d\tilde{t})^2 - \frac{1}{2} \{t, \tilde{t}\} (d\tilde{t})^2 \right) = \left(\frac{2}{3} \Lambda(t) - \frac{1}{2} \{\tilde{t}, t\} \right) (dt)^2. \quad (28)$$

The similarity of (26) to the formula for the transformation of the energy momentum tensor $T(z)$ under a holomorphic transformation $z \rightarrow \tilde{z} = \tilde{z}(z)$

$$\tilde{T}(\tilde{z}) (d\tilde{z})^2 = \left(T(z) - \frac{c}{12} \{\tilde{z}, z\} \right) (dz)^2, \quad (29)$$

will not be lost on readers familiar with two-dimensional Euclidean Conformal Quantum Field Theories. Similar formulae have arisen in the closely related context of calculations of the entropy of branes [25].

4 Covariant formulation

The concept of a Schwarzian derivative, thought of as a quadratic differential has been generalised by Osgood and Stowe in real dimension greater than two to a Schwarzian Tensor [26].

If $\{M, g, \phi\}$ is an n -dimensional (pseudo-)Riemannian manifold $\{M, g\}$ and ϕ a real valued function on M , then they define the Schwarzian tensor $B_{\mu\nu}(\phi)$ of ϕ to be the symmetric, trace-free second rank covariant tensor

$${}_g B_{\mu\nu}(\phi) = \nabla_\mu \nabla_\nu \phi - \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{n} \left(\nabla^2 \phi - (\nabla \phi)^2 \right) g_{\mu\nu}. \quad (30)$$

If $f : \{M, g\} \rightarrow \{\tilde{M}, \tilde{g}\}$ is a *conformal transformation*, that is such that the pull-back of \tilde{g} to M is a *Weyl conformal rescaling* of $g : f^*\tilde{g} = e^{2\phi}g$, then taking $\phi = \ln(|df|)$ the *Schwarzian derivative* of f , written Sf , is defined to be

$$Sf_{\mu\nu} = gB_{\mu\nu}(\phi). \quad (31)$$

If $n = 2$ this reduces to the standard definition because g is the Euclidean metric and $z = x + iy$, then

$$Sf_{\mu\nu} = B_{\mu\nu}(\ln|\frac{df}{dz}|) = \begin{pmatrix} \Re Sf & -\Im Sf \\ -\Im Sf & -\Re Sf \end{pmatrix}_{\mu\nu}, \quad (32)$$

so that

$$Sf_{\mu\nu}dx^\mu dx^\nu = \Re(Sf)(dx^2 - dy^2) - 2\Im(Sf)dxdy = \Re(Sf)dz^2. \quad (33)$$

The Schwarzian tensor also behaves nicely under composition of conformal transformations:

$$gB_{\mu\nu}(\phi + \sigma) = gB_{\mu\nu}(\phi) + \tilde{g}B_{\mu\nu}(\sigma). \quad (34)$$

Moreover if

$$\{M, g\} \xrightarrow{h} \{\tilde{M}, \tilde{g}\} \xrightarrow{f} \{\tilde{\tilde{M}}, \tilde{\tilde{g}}\} \quad (35)$$

then

$$S(f \circ h) = h^*S(f) + S(h). \quad (36)$$

For our purposes, most important property of the Schwarzian tensor proved in [26] is that under a Weyl conformal rescaling

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\phi}g_{\mu\nu} \quad (37)$$

the trace-free Ricci tensor

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{n}Rg_{\mu\nu} \quad (38)$$

changes by a shift of a multiple of the the Schwarzian tensor

$$\tilde{S}_{\mu\nu} = S_{\mu\nu} - (n-2)gB_{\mu\nu}(\phi). \quad (39)$$

Now suppose the trace free Ricci tensor $\tilde{S}_{\mu\nu}$ vanishes. Then

$$\tilde{R}_{\mu\nu} = \frac{1}{n}R\tilde{g}_{\mu\nu} \quad (40)$$

The Ricci identity $\tilde{\nabla}^\mu(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}) = 0$, implies that

$$\tilde{R}_{\mu\nu} = \Lambda\tilde{g}_{\mu\nu}, \quad (41)$$

where Λ is a constant. Thus if one is able to solve the equation

$$S_{\mu\nu} = (n-2)gB_{\mu\nu}(\phi), \quad (42)$$

for ϕ , one may pass by a Weyl conformal rescaling to an Einstein metric for which the density of dark energy is constant. Eliminating $S_{\mu\nu}$ altogether is a rather strong condition (see [27, 28, 29, 30] for some results on this) and in what follows we shall take a less restrictive but more explicit approach using Friedmann-Lemaitre metrics. We conclude this section by noting *en passant* that there is a relationship between the Schwarzian tensor and the family of e Bakry-Émery-Ricci tensors recently introduced into scalar-tensor theory by Woolgar [31].

5 Friedmann-Lemaitre Models

We now apply the theory above to the case of Friedmann and Lemaitre's cosmic expansion.

5.1 Newtonian treatment

At the Newtonian level [17, 18, 19] we make the homothetic ansatz

$$\mathbf{x}_a = a(t)\mathbf{r}_a \quad (43)$$

where \mathbf{r}_a are independent of time, and constitute a central configuration [32] and $a(t)$ is the scale factor satisfying the Raychaudhuri equation but now with time independent Newton's constant and cosmological constant:

$$\frac{1}{a(t)} \frac{d^2 a(t)}{dt^2} = -\frac{GM}{a^3(t)} + \frac{\Lambda}{3} \quad (44)$$

Note that (44) is *identical* to the Raychaudhuri equation resulting from the applying the Einstein equations to a Friedmann-Lemaitre spacetime containing just pressure-free matter and a constant cosmological term Λ (the so-called Λ CDM model[19]. Making the temporal reparameterisation (5) and the replacement

$$a(t) = \sqrt{f'}\tilde{a} \quad (45)$$

we obtain

$$\frac{1}{\tilde{a}(t)} \frac{d^2 \tilde{a}(t)}{d\tilde{t}^2} = -\frac{\tilde{G}\tilde{M}}{\tilde{a}^3(t)} + \frac{\tilde{\Lambda}}{3} \quad (46)$$

where \tilde{G} and $\tilde{\Lambda}$ are again given by (16) and (17) .

5.2 General Relativistic treatment

At the level of the Einstein equations for Friedmann-Lemaitre metrics we have in Einstein conformal frame

$$ds_E^2 = -dt^2 + a^2(t)d\omega_k^2 \quad (47)$$

where t is Einstein cosmic time, $a(t)$ the scale-factor and $d\omega_k^2$ the metric on a three- space of constant curvature $k = 1, 0, -1$. If Λ and G are truly constant then The Einstein equations imply

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \quad (48)$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \quad (49)$$

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + P) = 0 \quad (50)$$

where ρ is the energy density of the matter and P its pressure. Of course if $P = 0$, as in the Λ CDM model, then $\rho = \frac{\rho_0}{a^3}$ and we recover (44) with $\tilde{M} = 4\pi\rho_0$. Note that in general (48,49) and (50) are not independent: given any two, the third follows as long as $\dot{a} \neq 0$.

Since in the case of the the Λ CDM model the equations are the same, the formulae for the change under the temporal reparamterization (??) are the same as the Newtonian case.

5.3 Conformal Transformations

On the other hand, if we re-parameterise the cosmic time coordinate t in (47) according to (5), in order that we have a Weyl conformal rescaling as in (2) we need to make the replacement

$$a(t) = f'\tilde{a}(\tilde{t}). \quad (51)$$

As noted in §2 It is clear that (45) and (51) not the same. In the one case we preserve the non-relativistic equations of motion which in the Friedmann-Lemaître metrics amount to the Raychaudhuri equation (44). In the other, we preserve the velocity of light. In other words if we were to pass from cosmic time, which is usually taken to coincide with atomic time, to another time, for example to astronomical or to De-Sitter time, we should also have a variable speed of light theory.

Following the discussion in §?? we put

$$ds_E^2 = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})d\omega_k^2 = e^{2\phi}\left\{-dt^2 + a^2(t)d\omega_k^2\right\} \quad (52)$$

with

$$a = e^{-\phi}\tilde{a}, \quad dt = e^{-\phi}d\tilde{t}. \quad (53)$$

Then

$$\frac{\dot{a}}{a} = e^{\phi}\left(\frac{\tilde{a}'}{\tilde{a}} - \phi'\right) \quad (54)$$

Thus the Friedmann equation (49) becomes

$$\left(\frac{\tilde{a}'}{\tilde{a}} - \phi'\right)^2 + \frac{k}{\tilde{a}^2} = \frac{8\pi G}{3}e^{-2\phi}\rho + e^{-2\phi}\frac{\Lambda}{3} \quad (55)$$

and (50) becomes

$$\rho' + 3(\rho + P)\left(\frac{\tilde{a}'}{\tilde{a}} - \phi'\right) = 0. \quad (56)$$

The last term in the modified Friedmann equation (55) may be understood from the identity

$$\Lambda\sqrt{-g}dt d^3x = e^{-2\phi}\Lambda\sqrt{-\tilde{g}}d\tilde{t}d^3x. \quad (57)$$

It is how one might naively expect the cosmological constant to transform from the way it appears in the action (1).

Under (53) The Raychaudhuri equation (48) equation becomes

$$\frac{\tilde{a}''}{\tilde{a}} - \frac{1}{\tilde{a}}(\tilde{a}\phi')' = -\frac{4\pi G}{3}e^{-2\phi}(\rho + 3P). \quad (58)$$

Thus we obtain

$$\frac{\tilde{a}''}{\tilde{a}} = -\frac{4\pi G}{3}(\tilde{\rho} + 3\tilde{P}) \quad (59)$$

with

$$\frac{1}{\tilde{a}}(\tilde{a}\phi')' - \frac{4\pi G}{3}e^{-2\phi}(\rho + 3P) = -\frac{4\pi G}{3}(\tilde{\rho} + 3\tilde{P}). \quad (60)$$

That is

$$\frac{1}{a}\frac{d}{dt}\left(a\frac{d\phi}{dt}\right) - \frac{4\pi G}{3}(\rho + 3P) = -\frac{4\pi G}{3}e^{2\phi}(\tilde{\rho} + 3\tilde{P}). \quad (61)$$

In effect the dark energy density deduced from the cosmic acceleration using the metric \tilde{g} would correspond to a cosmological term

$$\tilde{\Lambda} = -4\pi G(\tilde{\rho} + 3\tilde{P}) \quad (62)$$

and so if one is able to solve the second order non-linear differential equation (61) for ϕ given $\tilde{\rho} + 3\tilde{P}$, one may find a conformal frame for any choice of $\tilde{\Lambda}$.

5.4 Cosmological Scalars

In discussing Friedmann-Lemaitre metrics one often introduces various cosmological scalars which are invariant under rescaling the scale factor $a(t)$ and the cosmic time, t by independent constant factors [33]. Among them are the Hubble constant, deceleration and jerk given respectively by

$$H = \frac{1}{a}\frac{da}{dt}, \quad q = -a\left(\frac{da}{dt}\right)^{-2}\frac{d^2a}{dt^2}, \quad j = a^2\left(\frac{da}{dt}\right)^{-3}\frac{d^3a}{dt^3}. \quad (63)$$

All current cosmological observations are consistent with the so-called Λ CDM model which is equivalent to the statement that the jerk of the universes is one. For a recent discussion of direct measurements of the jerk and references to earlier literature see [34, 35].

We note here the relationship between the Schwarzian derivative to these three cosmological scalars

$$\{a, t\} = H^2(j - \frac{3}{2}q^2). \quad (64)$$

The Schwarzian derivative $\{a, t\}$ is invariant under a constant rescaling of the scale factor $a(t)$ but not of the cosmic time. By contrast the quantity

$$\frac{1}{(\frac{da}{dt})^2}\{a, t\} = \frac{1}{a^2}(j - \frac{3}{2}q^2) \quad (65)$$

is invariant under Moebius transformations (including constant rescaling) of the cosmic time coordinate t .

6 Conclusion

In this paper I have shown how the Schwarzian derivative enters the formula for the change of the density of dark energy under temporal re-parameterisations and how the Schwarzian tensor enters when considering conformal rescalings of the metric. I have illustrated this by considering a Λ CDM cosmology. It is striking that a similar behaviour crops up in the change of the stress tensor of a two-dimensional CFT under conformal transformations. This seems to hint at a deeper connection between dark energy and CFT's in 3+1 spacetime dimensions. In this connection it would be interesting to see whether or how the Schwarzian derivative enters the transformation formulae for the stress tensor.

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